Measurement Operator for Angular Dependent Photon Propagation in Contact-Free Optical Tomography

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ABSTRACT

Based on light propagation theory, the measurements of a contact-free imaging system with generalized optical components can be obtained from a linear transformation of the light intensity distribution on the surface of the imaging object. In this work, we derived the linear measurement operator needed to perform this transformation. Numerical experiments were designed and conducted for validation.

Keywords: Measurement Operator, Contact-Free, Diffuse Optical Tomography, Equation of Radiative Transfer

1 INTRODUCTION

Diffuse optical tomography (DOT) has attracted more and more interest recently due to its non-ionizing radiation, ability to detect functional physiological parameters, real time imaging and low instrumentation cost [1, 2]. Thus it has been widely applied in many clinical areas, such as breast imaging [3-5], joint imaging [6-9], brain imaging [1, 10], vascular imaging [11] and small animal imaging [12-14] in the past two decades. The basic concept for most DOT algorithms is to fit the measurements obtained by the detector with the outcome from a forward model. Generally, this can be written into an unconstrained optimization problem [15, 16]:

\[ \min_x \frac{1}{2} \| Q \cdot F(x) - Z \|^2 + R(x) \]  

(1)

Where \( x \) is the optical property that needs to be reconstructed, \( u \) is the light intensity distribution within the imaging object, \( Z \) is the observation from the detectors, \( F \) is the operator of the forward model, \( Q \) is the measurement operator and \( R(x) \) is regularization term.

Fiber-based DOT systems are commonly used for delivering source illumination and measuring the photons after they interact with the tissue of interest. Another popular type of system employs a CCD camera at the detection side, replacing fibers as detectors. Fiber based detectors are often physically in contact with the surface of the tissue being probed, while CCD camera detectors are typically not in contact with the tissue. Thus, CCD camera based systems are referred to as “contact-free” systems. Fiber based systems usually have simpler measurement operator \( Q \) (i.e. equivalent to the partial current operator), but can be cumbersome to use when handling complex geometries and may introduce higher measurement noise. To overcome these difficulties and achieve better performance, contact-free imaging systems are becoming more popular in DOT. For these imaging systems the measurement operator \( Q \) is crucial and not as straightforward compared to the fiber systems. Light propagation from the object’s surface to the CCD chip (image plane) need to be modeled to construct \( Q \). To date several algorithms to handle this problem have already been proposed. A Monte Carlo based simulation method [17] can handle this problem. However, usually the CPU time to obtain a
required accuracy is unacceptably long. In 2003, Ripoll et al proposed an efficient free-space diffuse light transport model [18]. In the same year, Schulz et al. proposed a simplified model [19] with a pinhole approximation based on Ripoll’s model. In 2009, X. Chen et al. put forward an improved model [20] based on the hybrid radioisotropy-radiance theorem. However, all of these frameworks are based on diffuse photons approximation, so they lack the ability of handling the angular dependency of light intensity in transport-theory-based model. Moreover, the pinhole assumption in Ripoll’s and Schulz’s work makes their models only suitable for conditions of small aperture.

In this work, we generalized the pinhole assumption to a semi-pinhole assumption. That is we apply the pinhole assumption to locate the source point but compute an integral over a specific solid angle domain to achieve the energy received for each pixel on the image plane. Based on this assumption we present the first measurement operator based on transport theory for contact-free systems. Unlike diffusion theory, transport theory provides information about the angular dependence of light propagation. Furthermore, we also take general optical system components (i.e. lenses and mirrors) into account, which is often applied to acquire or amplify the signals from certain areas or certain directions. The basic theory, formulations, and numerical results of transport theory are presented. Our results demonstrate the validity and accuracy of the transport theory based measurement operator.

2 THEORY AND METHOD

2.1 Radiative transport equation based forward model

The most general deterministic forward model for light transport in tissue is the radiative transport equation (RTE) [21], which takes angular dependency of light intensity into account. The RTE can be obtained by considering the energy balance for a small control volume of the scattering dominant medium. In words, the RTE is an energy balance equation for a control volume. It models the storage of energy that comes from energy entering through the control volume surface, energy generated within the control volume, and subtracts energy dissipated within the control volume. This equation describes the behavior of light intensity \( I(\mathbf{r}, \mathbf{s}) \), with units of W/mm\(^2\)/sr, that is, the energy shift in direction \( \mathbf{s} \), per unit solid angle, per unit area normal to \( \mathbf{s} \), per unit time. The time dependent RTE is:

\[
\frac{1}{c} \frac{\partial I(\mathbf{r}, \mathbf{s})}{\partial t} + \nabla \cdot \mathbf{s} I(\mathbf{r}, \mathbf{s}) + (\mu_a(\mathbf{r}) + \mu_s(\mathbf{r})) I(\mathbf{r}, \mathbf{s}) = \frac{\mu_s(\mathbf{r})}{4\pi} \int_{4\pi} I(\mathbf{r}, \mathbf{s}_i) p(\mathbf{s}, \mathbf{s}_i) d\Omega_i + q(\mathbf{r}, \mathbf{s}) \tag{2}
\]

![Figure 1. Reflection and refraction on the surface of the scattering medium.](image)

Where \( c \) is the light speed inside the medium, \( \mu_a(\mathbf{r}) \) and \( \mu_s(\mathbf{r}) \) are the absorption and scattering coefficients respectively. Both of them are functions of position vector \( \mathbf{r} \). Here, \( q(\mathbf{r}, \mathbf{s}) \) is the source term representing the spatial and angular distribution of the light source in units of W/mm\(^2\)/s, and \( p(\mathbf{s}, \mathbf{s}_i) \) is the phase function with the physical meaning the probability of a photon moving in the direction of \( \mathbf{s} \) to be scattered into direction \( \mathbf{s}_i \). The following reflective boundary condition is often employed with Eq. (2) to complete the forward light transport model:

\[
I(\mathbf{r}_b, \mathbf{s}) = \begin{cases} 
q(\mathbf{r}_b, \mathbf{s}) + R(\mathbf{s}_i) I(\mathbf{r}_b, \mathbf{s}), & \text{for } \mathbf{n} \cdot \mathbf{s} < 0 \\
(1 - R(\mathbf{s}_i)) I(\mathbf{r}_b, \mathbf{s}), & \text{for } \mathbf{n} \cdot \mathbf{s} \geq 0
\end{cases} \tag{3}
\]
As shown in Fig. 1, \( \mathbf{s}_1 \) and \( \mathbf{s} \) are a pair of incident direction and reflected direction with the surface normal \( \mathbf{n} \), we have
\[
\mathbf{s}_1 = \mathbf{s} - 2(\mathbf{s} \cdot \mathbf{n})\mathbf{n}
\]
according to Snell’s law. The reflectivity \( R(\mathbf{s}_1) \) is given by Fresnel’s law:
\[
R = \frac{1}{2} \left( \frac{\cos \theta_i - n \cos \theta_T}{\cos \theta_i + n \cos \theta_T} \right)^2 + \frac{1}{2} \left( \frac{\cos \theta_T - n \cos \theta_i}{\cos \theta_T + n \cos \theta_i} \right)^2
\]
Where \( n \) is the refractive index of the medium.
Thus the forward operator \( F \) is well defined by Eq. (2) along with boundary condition (3), with the input \( x = (\mu_0, \mu_s) \) and the outcome being the angular and space dependent radiance of the imaging object.

2.2 Surface radiation theory
According to the surface radiation theory [21], the unit emission power \( dP(\mathbf{r}, \mathbf{s}) \) [W] at \( \mathbf{r} \) with the solid angle \( \mathbf{s} \) can be described with the radiative light intensity \( I(\mathbf{r}, \mathbf{s})[\text{W/mm}^2/\text{sr}] \) given by:
\[
dP(\mathbf{r}, \mathbf{s}) = I(\mathbf{r}, \mathbf{s})\mathbf{n} \cdot \mathbf{s} d\Omega dA
\]
Then the total power transferred to the aperture of the CCD camera from the unit area \( dA \) centered at \( \mathbf{r} \) on the surface is given by:
\[
P(\mathbf{r}) = dA \int_{\Omega_A(\mathbf{r})} I(\mathbf{r}, \mathbf{s})\mathbf{n} \cdot \mathbf{s} d\Omega
\]
where \( \Omega_A(\mathbf{r}) \) is the solid angle set contains any solid angle starting at \( \mathbf{r} \) that travels through the optical components system, and finally points to the aperture of the CCD camera (Fig. 2).

![Figure 2. Light propagation from the object’s surface, through the optical system, to the aperture of the CCD camera.](image)

With the semi-pinhole and no loss of energy assumption, all the photons emitted from \( dA \) are received by a small unit area \( dA_D \) surrounding \( \mathbf{r}_D \) on the image plane, with \( \mathbf{r}_D \) determined by the ray path of the light emitted from \( \mathbf{r} \) as it passes through the optical center of the aperture. Then according to the conservation of energy law, we have the following mathematical expression for total power transferred:
\[
P(\mathbf{r}) = dA \int_{\Omega_A(\mathbf{r})} I(\mathbf{r}, \mathbf{s})\mathbf{n} \cdot \mathbf{s} d\Omega = dA_D Q(\mathbf{r}_D)
\]
Where \( Q(\mathbf{r}_D) \) is the quantity we need to obtain and it represents the energy received per unit area at \( \mathbf{r}_D \) on the image plane, with the unit \( \text{W/mm}^2 \).

2.3 Back-retracing, acceptance-rejection sampling and coordinate transformation
The formula for \( Q(\mathbf{r}_D) \) can be given by a slight modification of Eq. (6):
\[ Q(\vec{r}_D) = \frac{\vec{n}dA}{dA_D} \cdot \int_{\Omega_A(\vec{r})} l(\vec{r}, \vec{s}) \vec{s}d\Omega \]  

(7)

However, we need to know the source position \( \vec{r} \), the integral over domain of solid angle \( \Omega_A(\vec{r}) \) and the area ratio \( \frac{\vec{n}dA}{dA_D} \) on the right hand side of Eq. (7) to obtain the value of \( Q(\vec{r}_D) \).

The source position vector \( \vec{r} \) can be determined through a back-retracing method: we let photons emit from \( \vec{r}_D \) on the image plane, require the photons pass through the optical center of the aperture of the CCD camera, monitor the light ray generated by the photons, and locate the position \( \vec{r} \) that the photons first touch on the surface of the imaging medium. We use \( Tr(\vec{r}) = \vec{r}_D \) to denote the forward light propagation, where \( Tr \) is the light propagation operator from the object’s surface to the image plane, this leads to:

\[ \vec{r} = Tr^{-1}(\vec{r}_D) \]  

(8)

The integral over the domain of solid angle \( \Omega_A(\vec{r}) \) in Eq. (7) can be estimated with an acceptance-rejection sampling method [22]. We discretized a solid angle domain into \( N \) small solid angles \( \{\vec{s}_i\}_{i=1}^N \) with the weights \( \{d\Omega_i\}_{i=1}^N \), then we only collect those directions that can pass the photon into the aperture. The mathematical expression is:

\[ \int_{\Omega_A(\vec{r})} l(\vec{r}, \vec{s}) \vec{n} \cdot \vec{s}d\Omega \approx \sum_{i=1}^{N} Id(\vec{r}, \vec{s}_i) \rightarrow Aperture)l(\vec{r}, \vec{s}_i) \vec{n} \cdot \vec{s}_i d\Omega_i \]  

(9)

Where \( Id(x) \) is an indicator function that returns 1 if condition \( x \) is true, otherwise it returns 0.

To obtain the area ratio, we give both \( \vec{r} \) and \( \vec{r}_D \) local parameterized coordinates as \( \vec{r}(u, v) \) and \( \vec{r}_D(x, y) \). With this expression, through a coordinates transform, we have

\[ \frac{\vec{n}dA}{dA_D} = \vec{r}_u \times \vec{r}_v |\frac{\partial(u, v)}{\partial(x, y)}| \]  

(10)

The absolute value of the Jacobian matrix \( |\frac{\partial(u, v)}{\partial(x, y)}| \) can be estimate with a perturbation method.

An expression for \( Q(\vec{r}_D) \) is obtained by plugging Eq. (8)-(10) into the Eq. (7).

To summarize, the algorithm for modeling the angular dependent light propagation is as follows:

**Algorithm** Angular dependent light propagation model

**Assumption:**
- Semi-pinhole assumption for CCD camera’s aperture.
- No loss of energy during the propagation.

**Steps:**
1. Discretize the CCD chip equally with \( \{\vec{r}_D(x_1, y_1), ..., \vec{r}_D(x_K, y_K)\} \).
2. Discretize the solid angle domain with \( \{(\vec{s}_1, d\Omega_1), (\vec{s}_2, d\Omega_2), \cdots, (\vec{s}_N, d\Omega_N)\} \).
3. for \( k = 1 \) to \( K \) do
   4. \( Q_k = 0 \)
   5. \( \vec{r}_D = \vec{r}_D(x_k, y_k) \)
   6. Using perturbation method to estimate \( f = |\frac{\partial(u, v)}{\partial(x, y)}| \) at \( \vec{r}_D(x_k, y_k) \) in Eq. (10)
   7. for \( i = 1 \) to \( N \) do
8. \[ \text{if } \text{Id}(\vec{r}, \vec{s}_k) \rightarrow \text{Aperture} = 1 \]
9. \[ Q_l = Q_l + \int (\vec{r}, \vec{s}_i) \vec{n} \cdot \vec{s}_i d\Omega_i \]
10. \[ \text{end if} \]
11. \[ \text{end for} \]
12. \[ \text{end for} \]

3 NUMERICAL EXPERIMENT AND RESULTS

3.1 Problem setting and analytical solution

The algorithm was validated through a numerical experiment; the result was compared with the analytical solution. The experiment consisted of simulating light emission from an object, allowing the light to pass through a simulated optical system lens aperture, and travel to a simulated CCD camera chip (Fig. 3). We set a round thin plate with isotropic radiation and uniform light intensity distribution \( I \) perpendicular to the optical axis in front of the aperture, with a distance \( l_1 \), and adjust the image distance \( l_2 \) to make the image of the plate well focused. Then according to the Cosine Fourth Law [23], the analytical solution for \( Q(\vec{r}_D) \) with \( \vec{r}_D \) (Fig. 3) is given by:

\[
Q(\vec{r}_D) = C \cos^4 \theta_D
\]

(11)

Where \( C \) is a constant that can be determined by \( \theta, l, l_1 \) and \( l_2 \). In our experiment, the parameters of this system were set as \( l = 1, \theta = 0.341^\circ, l_1 = 1050, l_2 = 52.5 \).

3.2 Numerical results

We used the correlation factor \( \text{Corr}(Q^C, Q^A) \) as [24] to evaluate the performance of algorithm, where \( Q^C \) and \( Q^A \) are calculated power density from our algorithm and analytical solution from the Cosine Fourth Law, respectively. \( \bar{Q} \) and \( \sigma(Q) \) are the mean and variance of \( Q \) respectively. The \( \text{Corr}(Q^C, Q^A) \) value represents the correlation level between analytical solution \( Q^A \) and the calculated solution \( Q^C \). It ranges from \(-1\) to \(1\). We also considered relative error factor, given by:

\[
\text{err}(Q^C, Q^A) = \frac{\sum_{i=1}^{K} \text{Id}(Q^A_i \neq 0) \left| \frac{Q^C_i - Q^A_i}{Q^A_i} \right|}{\sum_{i=1}^{K} \text{Id}(Q^A_i \neq 0)}
\]
The $err(Q^C, Q^A)$ value represents the average relative error of all non-zero values. It ranges for 1 to $\infty$. Higher $Corr(Q^C, Q^A)$ and lower $err(Q^C, Q^A)$ indicate higher similarities between $Q^C$ and $Q^A$.

The numerical results are shown in Fig. 4. Figure 4(a) displays the calculated power density on the image plane with the settings described in Section 3.1. Figure 4(b) is the comparison of $Q^C$ and $Q^A$ on the cross-section $y=0$. The simulation results and the analytical results are a good match. Moreover, $Corr$ and $err$ with $Q^C$ and $Q^A$ were computed based on the formula above:

$$Corr(Q^C, Q^A) = 1 - 9.49 \times 10^{-6}, \quad err(Q^C, Q^A) = 1.87 \times 10^{-5}$$

![Figure 4. Result of numerical experiment. (a) The calculated power density on the image plane with the settings in Figure 3. (b) Comparison on the cross-section $y = 0$.](image)

4 CONCLUSION

In this work, we constructed the first measurement operator for RTE-based contact-free imaging system. Generalized optical components were also taken into account.

Instead of the pinhole assumption, we only require a more relaxed semi-pinhole assumption to build our model. With this assumption, the contribution of power density for each pixel on the image plane was decomposed into three steps:

1) Use the pinhole assumption to back-trace the source position.
2) Employ an acceptance-rejection sampling method to compute the contribution from the effective solid angle domain of the source.
3) Estimate the area ratio through a perturbation method, then, apply it as a correction factor to the result in Step 2.

To validate our algorithm, we conducted a numerical experiment with a known analytical solution $Q^A$. The simulation solution $Q^C$ was compared to $Q^A$. The accuracy of our model was shown through high correlation factor between the two solutions $Corr(Q^C, Q^A) = 1 - 9.49 \times 10^{-6}$, and low average relative error of all non-zero values between the solutions $err(Q^C, Q^A) = 1.87 \times 10^{-5}$.

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REFERENCE